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CASIMIR INTERACTION OF COSMIC STRINGS: MASSIVE FIELD

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Abstract. Within the tr ln-formalism we study the influence of quantized field on the vacuum interaction of cosmic strings. We consider the real-valued massive scalar field with minimal coupling. It is shown that at the inter-string distances, which visibly exceed the Compton length $l_c = m^{-1}$, the appearance of mass leads to the exponential decay of the effect. Whereas at small with respect to l_c distances, but much larger than the string's width, the mass effect becomes insignificant; and the massive field contributes to the Casimir energy comparably with the massless field.

Keywords: *the Casimir effect, topological strings, the effective action, dimensional regularization*

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1. INTRODUCTION

The study of the evolution of the cosmic string network in the process of its formation during phase transitions in the early Universe involves the study of processes that occur during close flights of strings and during their collisions. In these situations, taking into account the vacuum (Casimir) interaction of strings can be significant [1, 2].

For the first time, an estimate of this effect for the case of parallel infinitely thin strings was obtained in [3]. Subsequently, the inaccuracy made there was corrected by various methods in the works [4-6]. In all these works, the authors limited themselves to the case of a massless field.

The Casimir interaction of strings is considered below in a relatively simple formulation, yet admitting an analytical solution. As in the works listed above, we will limit ourselves to looking at a static system of parallel infinitely thin strings and a real scalar field but we will not limit ourselves to the case of a field mass equal to zero.

The motivation for such a task statement is the following. First of all, we note that the energy of the vacuum interaction of strings per unit of length has the dimension of the square of the inverse length and of the dimensional quantities can depend only on the distance between the strings d, the radius of the strings a and the Compton length of the quantized field under consideration $l_c = \mathrm{m}^{-1}$. All the listed values are dimensionally dependent. But the Compton length for the heaviest currently known particle (t-quark) is $l_c \sim 10^{-15}$ cm, and the thickness of GUT strings is $a \sim 10^{-28}$ cm, which is many orders of magnitude less. In this case, the distance between the strings is $d \gg 2a$. In this situation, if we limit ourselves to the distances d, which are much greater than the thickness of the strings, then the strings can be considered as infinitely thin. Then the energy of interaction of two strings parallel to the z axis, attributed to the unit of string length, can depend only on d and the Compton length of the field under consideration and, therefore, can always be represented as

$$\frac{E_{cas}}{Z} = -\frac{4}{15\pi}\frac{\mu_1\mu_2}{d^2}\mathcal{F}\left(md\right), \qquad Z = \int dz, \tag{1}$$

where $\mu_{1,2}$ is the mass of the string per unit length, F is a real-valued function. The coefficient before F is determined for convenience considerations, it is further selected in such a way that it coincides with the energy of the Casimir interaction of infinitely thin strings in the case of a scalar field with minimal coupling at zero field mass.

Let's consider the behavior of the function $F(z)$ when $z = md$ tends to zero. This limit can be considered as a transition to the case of a massless field at finite values of d and, consequently, when we choose the coefficient in (1), this limit implies, $F = 1$. On the other hand, with equal bases, this limit can also be considered as the limiting transition $d \to 0$ at finite values of mass. Thus, the scale at which the mass effect will be significant is the Compton length, and at distances between strings smaller or of the order of the Compton length (but larger than the transverse size of the strings), the mass effect will not be significant and the partial contribution of massive modes to the energy of the vacuum interaction of strings will be comparable with the contribution of a massless field.

The work uses a system of units $G = \hbar = c = 1$ and a space-time metric with a signature (+,−,−,−).

2. BACKGROUND SPACE METRIC

Consider a four-dimensional spacetime, which is a direct product of a two-dimensional Minkowski space on a two-dimensional Riemann surface. As is known, in this case, by an appropriate choice of coordinates, the metric of the considered space-time can always be reduced to the form

$$ds^2 = dt^2 - dz^2 - e^{-\sigma(\mathbf{x})}\left(dx_1^2 + dx_2^2\right), \tag{2}$$

where $\mathbf{x} = (x_1, x_2)$.
Let

$$\sigma(\mathbf{x}) = \sum_a \sigma_a\left(\left|\mathbf{x} - \mathbf{x}_a\right|\right), \tag{3}$$

$$\left|\mathbf{x} - \mathbf{x}_a\right| = \left[\left(x_1 - x_{a1}\right)^2 + \left(x_2 - x_{a2}\right)^2\right]^{1/2},$$

where $\mathbf{x}_a$ is a set of fixed points. In this case, the scalar curvature has the form

$$R = \sum_a R_a = \sum_a e^{\sigma} \Delta_E \sigma_a, \tag{4}$$

where Δ_E is a two-dimensional Euclidean Laplacian. And if the supports of the partial contributions of $\Delta_E \sigma_a$ are compact and do not overlap, then we get an ultrastatic spacetime whose curvature in the plane $(x_1 x_2)$ is different from zero only in a set of non-overlapping compact neighborhoods of x_a.

Let's choose the functions σ_a in the form

$$\sigma_a = 2\left(1 - \beta_a\right)\ln\left|\mathbf{x} - \mathbf{x}_a\right|, \tag{5}$$

where $0 < \beta_a < 1$ for all a. As it was shown in [7], the metric obtained in this way is a solution to the Einstein equation, in the right part of which there is an energy-momentum tensor of the form

$$\begin{gathered} T_{\mu\nu}\left(t, z, \mathbf{x}\right) = \\ = e^{\sigma(\mathbf{x})} \sum_a \mu_a \delta^2\left(\mathbf{x} - \mathbf{x}_a\right) \mathrm{diag}\left(1,1,0,0\right), \\ \mu_a = \frac{1 - \beta_a}{4}, \end{gathered} \tag{6}$$

and the corresponding solution corresponds to the space-time of a system of parallel infinitely thin cosmic strings. In this case, the two-dimensional surface $(x_1 x_2)$ is a locally flat hypersurface with a set of conical features localized at points x_a, and the parameter μ_a makes sense of the linear energy density of the a-th string and determines the angle deficit associated with the a-th conical feature

$$\delta\varphi_a = 8\pi\mu_a = 2\pi\left(1 - \beta_a\right).$$

In the case of single infinitely thin string, the features of space-time are the absence of any dimensional parameters in the metric and a high degree of symmetry. The former makes it possible to assert that in the case of a massless field, the vacuum mean of the energy-momentum tensor operator can depend only on the distance to the singularity and in four space-time measurements

$$\left\langle T_{\mu\nu} \right\rangle_{vac}^{ren} \sim r^{-4}.$$

The latter makes it possible to separate the variables in the field equation, construct an analytical expression for the corresponding Green's function, and calculate the renormalized vacuum mean of the energy-momentum tensor operator [8-12]. In the case of two or more strings and massive fields, the latter is not possible and forces the use of methods of perturbation theory [4-6]. At the same time, the ability to work within the framework of perturbation theory is provided by the smallness of the parameters (1−β). It is assumed that for cosmic strings considered within the framework of the Grand Union Theory, the value of the parameters (1 − β) is of the order 10^{-6}.

3. VACUUM INTERACTION

The case of a massive real-valued scalar field φ corresponds to the choice of action in the form of

$$S_\varphi = -\frac{1}{2}\int d^4x \phi\left(x\right) L\left(x, \partial\right) \phi\left(x\right),$$

where the field operator

$$L\left(x, \partial\right) = \sqrt{-g}\left(\Box + m^2\right),$$

$$\Box = \nabla_\mu \nabla^\mu$$

is the Laplace-Beltrami operator.

We limited ourselves to the case of a scalar field with minimal coupling. This is due to the fact that in the case of metric (2) with a conformal factor (5) considered below, a non-minimal relationship leads to the appearance of a potential field in the equation with δ-shaped features. The appearance of such potential features requires a separate consideration and the results of calculations may differ depending on how such features are interpreted [13, 14].

Let's represent the operator $L(x, \partial)$ as

$$L(x,\partial) = \left(\partial^2 + m^2\right) + \delta L(x,\partial), \tag{7}$$

$$\partial^2 = \partial_t^2 - \partial_1^2 - \partial_2^2 - \partial_z^2.$$

Here and further, scalar products of 4-vectors are understood in the sense of the metric of the Minkowski space. In this case, the operator $\delta L(x, \partial)$ corresponding to metric (2) has the form

$$\delta L(x,\partial) = \Lambda(\mathbf{x})\left(\partial_t^2 - \partial_z^2 + m^2\right) \tag{8}$$

$$\Lambda(\mathbf{x}) = e^{-\sigma(\mathbf{x})} - 1.$$

The value that is often used in the study of vacuum energy is the effective action. In the Schwinger–DeWitt approach, it can be represented as

$$W_{eff} = \frac{i}{2}\mathrm{tr}\ln L = \frac{i}{2}\ln\det L,$$

where L is understood as an operator in an abstract Hilbert space, where the basis vectors $|x\rangle$ are eigenvectors of a commuting set of Hermitian operators $\hat{x}^\mu$ with conditions of normalization

$$\langle x \mid x' \rangle = \delta^{(4)}(x - x')$$

and completeness

$$\sum_x |x\rangle\langle x| = \mathbf{1}.$$

In this case, the trace of the operator is defined as

$$trQ = \int d^4x \langle x|Q|x\rangle,$$

and in the coordinate representation, the matrix element has the form

$$\langle x|L|x'\rangle = L(x,\partial_x)\delta^{(4)}(x - x'),$$

see [15-17].

The trace defined in this way during the calculation makes it possible to move to another orthonormal basis, as which we will choose the Fourier basis.

Further, it is known that in the case when external factors (metric, boundaries, external fields, etc.) do not explicitly depend on time, the effective action of W_{eff} is proportional to the total vacuum energy of $\mathcal{E}_{vac}$, namely:

$$W_{eff} = -T\mathcal{E}_{vac},$$

where T is the total time [18] (see also [19]) and, consequently, within the framework of the tr ln-formalism

$$\mathcal{E}_{vac} = -\frac{i}{2T}\ln\det L. \tag{9}$$

If the operator δL, included in (7) can be considered as a small perturbation, then we have

$$\begin{aligned} \ln\det L &= \ln\det\left(\partial^2 + m^2 + \delta L\right) = \\ &= \ln\det\left(\partial^2 + m^2\right) + \ln\det\left[1 + \left(\partial^2 + m^2\right)^{-1}\delta L\right] = \\ &= \mathrm{tr}\ln\left(\partial^2 + m^2\right) + \mathrm{tr}\ln\left[1 + \left(\partial^2 + m^2\right)^{-1}\delta L\right] = \\ &= \mathrm{tr}\ln\left(\partial^2 + m^2\right) + \mathrm{tr}\left[\left(\partial^2 + m^2\right)^{-1}\delta L\right] - \\ &- \frac{1}{2}\mathrm{tr}\left[\left(\partial^2 + m^2\right)^{-1}\delta L\left(\partial^2 + m^2\right)^{-1}\delta L\right] + \ldots \end{aligned} \tag{10}$$

However, the resulting formal expression is well defined only if the operators included in it are operators with a trace [20]. In our case, this is not the case, and when calculating traces, regularization will be required, as which we will choose dimensional regularization.

In the Fourier basis, the traces available in (10) are reduced to the standard expressions for quantum field theory. In particular, the first two terms of (10) in the framework of the dimensional regularization method are reduced to the expression

$$\begin{aligned} &\mathrm{tr}\ln\left(\partial^2 + m^2\right) + \mathrm{tr}\left(\left(\partial^2 + m^2\right)^{-1}\delta L\right) = \\ &= -iTZm^D\frac{\Gamma[-D/2]}{(4\pi)^{D/2}}\int\left(\Lambda(\mathbf{x}) + 1\right)d^2x = \\ &= -iTZm^D\frac{\Gamma[-D/2]}{(4\pi)^{D/2}}\int\sqrt{-g(\mathbf{x})}\,d^2x, \end{aligned} \tag{11}$$

$$D = 4 - 2\varepsilon.$$

The corresponding contribution to the effective action coincides with the first term of the Schwinger–DeWitt expansion and is discarded during renormalization [17].

Thus, in order to isolate the Casimir contribution to the total vacuum energy in the first non-decreasing order of perturbation theory, we must limit ourselves to the third term of decomposition (10):

$$\mathcal{E}_{vac} = \frac{i}{4T}\mathrm{tr}\left(\left(\partial^2 + m^2\right)^{-1}\delta L\left(\partial^2 + m^2\right)^{-1}\delta L\right). \quad (12)$$

In the Fourier basis, this expression takes the form

$$\mathcal{E}_{vac} = \frac{i}{4T}\int\frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{\delta L\left(k, i(p+k)\right)\delta L(-k, ip)}{\left[p^2 - m^2\right]\left[(p+k)^2 - m^2\right]}, \quad (13)$$

where

$$\delta L(k, ip) = \int d^4x e^{ikx}\left[\delta L(x, \partial)\,|_{\partial\to -ip}\right]. \quad (14)$$

In our case, from (8) we get

$$\delta L(k, ip) = -\Lambda(k)\left(p_0^2 - p_z^2 - m^2\right) \quad (15)$$

and thus, the vacuum energy is determined by the expression

$$\mathcal{E}_{vac} = \frac{i}{4T}\int\frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\times \frac{\left(p_0^2 - p_z^2 - m^2\right)^2}{\left[p^2 - m^2\right]\left[(p+k)^2 - m^2\right]}\times \Lambda(k)\Lambda(-k). \quad (16)$$

When obtaining expression (16), it was taken into account that

$$\Lambda(k) = 4\pi^2\delta\left(k^0\right)\delta\left(k^z\right)\Lambda(\mathbf{k}), \quad (17)$$

where $\Lambda(\mathbf{k})$ is two-dimensional Fourier image of the function $\Lambda(\mathbf{x})$, $\mathbf{k} = (k^1, k^2)$. Therefore, $k^0 = k^z = 0$.

The d^4p integral in expression (16) diverges, but has the standard form for the dimensional regularization method.

Wick rotation

$$p^0 = ip_E^0, \qquad d^4p = id^4p_E, \qquad p^2 = -p_E^2$$

and further replacement of d^4p by $\tilde{\mu}^{4-D}d^Dp_E$, $D = 4 - 2\varepsilon$, bring expression (16) to the form

$$\mathcal{E}_{vac}^{reg} = -\frac{\tilde{\mu}^{4-D}}{4T}\int\frac{d^4k}{(2\pi)^4}\Lambda(k)\Lambda(-k)\times \int\frac{d^Dp_E}{(2\pi)^D}\frac{\left(p_0^2 + p_z^2 + m^2\right)_E^2}{\left(p^2 + m^2\right)_E\left[(p+k)_E^2 + m^2\right]}, \quad (18)$$

where $\tilde{\mu}$ is an arbitrary scale with a dimension of mass, which is introduced to preserve the dimension of the regularized expression (18).

The internal d^Dp_E integral has a typical form for quantum field theory and is calculated using Feynman parameterization (see, for example, [21]). In the subsequent d^4k integration, we will face the fact that the integrand contains the square $\Lambda(k)$ (17), i.e. the squares $\delta(k^0)$ an $\delta(k^z)$. We deal with them in the standard way:

$$\left[\delta\left(k^0\right)\right]^2 = \delta\left(k^0\right)\delta(0) = \frac{\delta\left(k^0\right)}{2\pi}\int e^{ik^0t}dt\,|_{k^0=0} = \frac{T}{2\pi}\delta\left(k^0\right).$$

Similarly, for the integration of k^z:

$$\left[\delta\left(k^z\right)\right]^2 = \frac{Z}{2\pi}\delta\left(k^z\right).$$

As a result, for the regularized vacuum energy, we get

$$\mathcal{E}_{vac}^{reg} = -\frac{Z}{4(4\pi)^{D/2}}\int\frac{d^2k}{(2\pi)^2}\Lambda(\mathbf{k})\Lambda(-\mathbf{k})\times \int_0^1 d\alpha\left[2\Gamma(-2+\varepsilon)\Delta^2 + 2m^2\Gamma(-1+\varepsilon)\Delta + m^4\Gamma(\varepsilon)\right]\times \left(\frac{\Delta}{\tilde{\mu}^2}\right)^{-\varepsilon}, \quad (19)$$

where

$$\Delta = \alpha(1-\alpha)\mathbf{k}^2 + m^2.$$

Then, decomposing $\left(\Delta/\tilde{\mu}^2\right)^{-\varepsilon}$ in a small ε,

$$\left(\frac{\Delta}{\tilde{\mu}^2}\right)^{-\varepsilon} = 1 - \varepsilon\ln\frac{\Delta}{\tilde{\mu}^2} + \mathcal{O}\left(\varepsilon^2\right), \quad (20)$$

and discarding divergent members when removing the regularization, we get

$$\begin{aligned}\mathcal{E}_{vac}^{ren} &= \frac{Z}{4(4\pi)^2}\int\frac{d^2k}{(2\pi)^2}\Lambda(\mathbf{k})\Lambda(-\mathbf{k})\times \\ &\times\int_0^1 d\alpha\left(\Delta - m^2\right)^2\ln\frac{\Delta}{\tilde{\mu}^2} = \\ &= \frac{Z}{4(4\pi)^2}\int\frac{d^2k}{(2\pi)^2}\Lambda(\mathbf{k})\Lambda(-\mathbf{k})|\mathbf{k}|^4\times \\ &\times\int_0^1 d\alpha\alpha^2(1-\alpha)^2\ln\frac{\Delta}{\tilde{\mu}^2}.\end{aligned} \tag{21}$$

Assuming that the exponent σ in (8) is small and the substitution is fair

$$\Lambda(\mathbf{x}) \to -\sum_a \sigma_a(\mathbf{x}),$$

we come to the expression

$$\begin{aligned}\mathcal{E}_{vac}^{ren} &= \frac{Z}{4(4\pi)^2}\sum_{a,b}\int\frac{d^2k}{(2\pi)^2}\sigma_a(\mathbf{k})\sigma_b(-\mathbf{k})|\mathbf{k}|^4\times \\ &\times\int_0^1 d\alpha\alpha^2(1-\alpha)^2\ln\frac{\Delta}{\tilde{\mu}^2},\end{aligned} \tag{22}$$

where, when choosing σ_a in the form of (5), the Fourier image of the partial conformal factor is equal

$$\sigma_a(\mathbf{k}) = -\frac{16\pi\mu_a}{\mathbf{k}^2}e^{i\mathbf{k}\cdot\mathbf{x}_a}. \tag{23}$$

We see that the terms of the sum with $a \neq b$ correspond to the Casimir (depending on the relative distances between the strings) in (22), and under the assumption made, the Casimir interaction between the strings can be approximately considered as a pair. Therefore, it is enough to limit yourself to two parallel strings, located at a distance d from each other. At the same time, integration over α carried out in (19), leads the expression for the Casimir energy to the form

$$\begin{aligned}\mathcal{E}_{cas} &= \frac{8Z\mu_1\mu_2}{15}\int\frac{d^2k}{(2\pi)^2}e^{i\mathbf{k}\cdot\mathbf{d}}\times \\ &\times\left[\ln\frac{m}{\tilde{\mu}} + A(x)\left(1 - \frac{2}{x^2} + \frac{6}{x^4}\right) - \left(\frac{47}{60} - \frac{3}{2x^2} + \frac{6}{x^4}\right)\right],\end{aligned} \tag{24}$$

where

$$x = \frac{|\mathbf{k}|}{m}, \quad A(x) = \sqrt{1 + (2/x)^2}\,\mathrm{Arsh}\frac{x}{2}.$$

Thus, the further transformation of the expression (24) is reduced to calculating the two-dimensional Fourier integral of a rather cumbersome expression. Understood in the sense of generalized functions, the Fourier images of the terms of the integral expression that do not contain the function $A(x)$ are known [22]:

$$\int d^2k\frac{e^{i\mathbf{k}\cdot\mathbf{d}}}{|\mathbf{k}|^{2\lambda}} = \frac{(-1)^\lambda\pi}{2^{2\lambda-3}\Gamma^2(\lambda)}|\mathbf{d}|^{2(\lambda-1)}\ln|\mathbf{d}|, \qquad \lambda \in \mathbb{N}. \tag{25}$$

For $\lambda = 0$ the result is proportional to $\delta^2(\mathbf{d})$ and, consequently, is zero.

The remaining integrals have the form

$$c_n = \int\frac{d^2k}{(2\pi)^2}\frac{e^{i\mathbf{k}\cdot\mathbf{d}}}{|\mathbf{k}|^{2n}}A(|\mathbf{k}|/m), \quad n = 0,1,2. \tag{26}$$

Since these integrals represent Fourier images of the cylindrically symmetric functions of the variable $k = |\mathbf{k}|$, the transformation result will be a cylindrically symmetric function of the variable $d = |\mathbf{d}|$. This makes it possible to perform integration along the polar ang φ in the plane (k_1, k_2) using the integral [23]

$$\int_0^{2\pi} d\varphi e^{iqr\cos\varphi} = 2\pi J_0(qr). \tag{27}$$

But the remaining one-dimensional integrals by dk

$$c_n = \int\frac{dk}{2\pi}J_0(kd)k^{1-2n}A(k/m), \quad n = 0,1,2, \tag{28}$$

if we understand them as Riemann integrals, diverge either on the upper (c_0), or on the lower (c_1, c_2) limits.

The method we propose is to represent them as a sum of convergent Riemannian integrals and known Fourier images defined in terms of generalized functions.

To clarify the nature of the divergence of the integrals (28), we need to know the behavior of $A(x)$ for small and large values of the argument.

For small argument values it expands as

$$A(x) = 1 + \frac{1}{12}x^2 - \frac{1}{120}x^4 + \frac{1}{840}x^6 + \mathcal{O}(x^8), \tag{29}$$

while the asymptotic expansion ($x \gg 1$) is given by the expression

$$A(x) = \ln x + \frac{2\ln x + 1}{x^2} - \frac{2\ln x - 1/2}{x^4} + \mathcal{O}\left(\frac{\ln x}{x^6}\right). \tag{30}$$

Then, to regularize each of the c_n integrals, depending on the nature of the non-integrable feature, we subtract from the integrand and add the necessary number of corresponding expansion terms (counter-terms), making sure that the subtracted counter-terms make it possible to remove the convergence at the limit where it exists, but without a new divergence at another limit of integration, and so that the integral of the difference converges in the Riemann sense. It is clear that this makes sense only if the images of the counter-terms defined in the sense of generalized Fourier functions are known.

In this case, the subtracted counter-terms will regularize the non-integrable singularity of the integrand, and we get an integral expression well-defined as a Riemann integral, to which well-known, defined in the sense of generalized Fourier functions, images of individual counter-terms are added.

The peculiarity of the proposed procedure is that the subtracted counter-terms will be determined by the convergence of the one-dimensional integral (28), and we will carry out the corresponding identical transformation of subtraction-addition in the two—dimensional Fourier integral (26).

Applying the described procedure, we obtain the following expression for the Casimir energy:

$$\mathcal{E}_{cas} = \frac{4Z\mu_1\mu_2}{15\pi}\int_0^\infty dk\,kJ_0(kd)\times \left[\begin{matrix} A\left(\frac{k}{m}\right)\left(1-2\frac{m^2}{k^2}+6\frac{m^4}{k^4}\right)- \\ -\ln\frac{k}{m}+\frac{3}{2}\frac{m^2}{k^2}-6\frac{m^4}{k^4} \end{matrix}\right] - \frac{4Z\mu_1\mu_2}{15d^2}. \qquad (31)$$

It is noteworthy that the non-integrable term in (26) coincides with the known result for a massless scalar field. Thus, the dependence of the Casimir effect on mass, which interests us, is entirely determined by the integral term standing in (26), and for the function $\mathcal{F} = \mathcal{E}_{vac}(m) / \mathcal{E}_{vac}(0)$ formally introduced in (1), we obtain the clear expression

$$\mathcal{F} = -d^2\int_0^\infty dk\,kJ_0(kd)\times \left[A\left(1-2\frac{m^2}{k^2}+6\frac{m^4}{k^4}\right)-\ln\frac{k}{m}+\frac{3}{2}\frac{m^2}{k^2}-6\frac{m^4}{k^4}\right]+1, \qquad (32)$$

where the integral already converges as a Riemannian one.

After the replacement of the variable $s = k / 2m$ the integral is split into three:

$$\mathcal{F}(z) = 1 - z^2\left[h_0(z)-\frac{1}{2}h_1(z)+\frac{3}{8}h_2(z)\right], \quad z = 2md,$$

where $h_n(z)$ are defined as

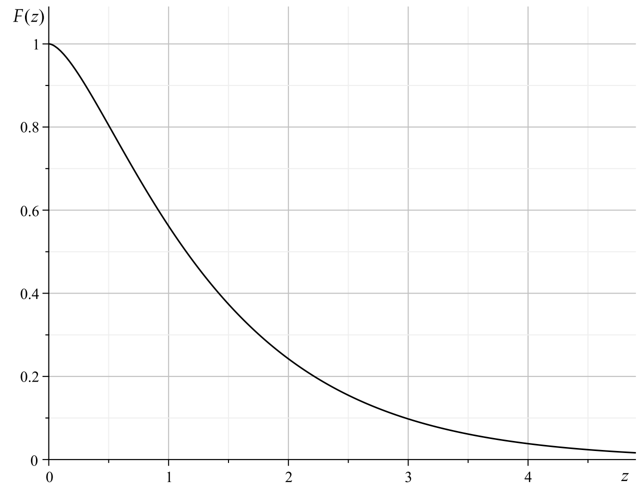

Fig. 1. Graph $\mathcal{F}(z)$

$$h_0(z) = \int_0^\infty ds J_0(sz)\left[\sqrt{1+s^2}\,\mathrm{Arsh}\, s - s\ln 2s\right],$$
$$h_1(z) = \int_0^\infty ds\frac{J_0(sz)}{s^2}\left[\sqrt{1+s^2}\,\mathrm{Arsh}\, s - s\right], \qquad (33)$$
$$h_2(z) = \int_0^\infty ds\frac{J_0(sz)}{s^4}\left[\sqrt{1+s^2}\,\mathrm{Arsh}\, s - s - \frac{s^3}{3}\right].$$

They are integrals to which regularized two-dimensional Fourier c_n integrals are reduced. These integrals can be calculated in the following form:

$$h_0(z) = \frac{1}{z^2}+\frac{1}{4}\left[K_0^2\left(\frac{z}{2}\right)-K_1^2\left(\frac{z}{2}\right)\right],$$
$$h_1(z) = \frac{z}{2}U(z), \qquad (34)$$
$$h_2(z) = -\frac{z^2}{9}\left[K_0^2\left(\frac{z}{2}\right)-K_1^2\left(\frac{z}{2}\right)\right]- -\frac{z^3}{18}U(z)-\frac{z}{6}K_0\left(\frac{z}{2}\right)K_1\left(\frac{z}{2}\right),$$

where $K_n(\cdot)$ is the Macdonald function, $U(\cdot)$ is the special Macdonald integral function of the following form:

$$U(z) = \int_z^\infty \frac{dx}{x^2}K_0^2\left(\frac{x}{2}\right), \qquad (35)$$

which can also be written using the Meijer G-function [24]:

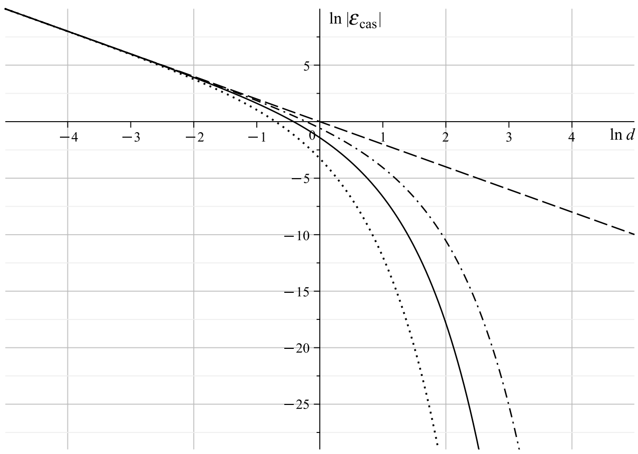

Fig. 2. The energy of the Casimir attraction of two strings as a function of distance (in units $l_c^{(m=1)} = 1$) in double logarithmic scaling: for massive fields with $m = 0.5$ (dashdotted curve), $m = 1$ (solid), $m = 2$ (dotted) and for a massless field (dashed, with a tangent of the angle of inclination to the horizontal −2)

$$U(z) = \frac{\sqrt{\pi}}{8} G_{1,3}^{3,0}\left(\left. \begin{matrix} 1 \\ -1/2, -1/2, -1/2 \end{matrix} \right| \frac{z^2}{4} \right). \quad (36)$$

As the result for the function $\mathcal{F}(z)$ we obtain

$$\mathcal{F}(z) = \frac{z^2}{4}\left[\begin{matrix} \left(\frac{z^2}{6} - 1\right)\left[K_0^2\left(\frac{z}{2}\right) - K_1^2\left(\frac{z}{2}\right)\right] + \\ + \left(\frac{z^2}{12} + 1\right) zU(z) + \frac{z}{4} K_0\left(\frac{z}{2}\right) K_1\left(\frac{z}{2}\right) \end{matrix} \right]. \quad (37)$$

The graph $\mathcal{F}(z)$ is shown in Fig. 1.

It follows from the resulting expression that when $z \gg 1$

$$\mathcal{F}(z) = \frac{\pi}{16} e^{-z}\left(15 - \frac{75}{2z} + \frac{25031}{128z^2} + \mathcal{O}(z^{-3})\right).$$

Thus, at distances greater than the Compton length of the massive field, the effect is suppressed exponentially.

In the opposite extreme case, when $z \ll 1$

$$\mathcal{F}(z) = 1 + \frac{5}{8} z^2\left(\ln\frac{z}{4} + \gamma + \frac{1}{3}\right) + \mathcal{O}\left(z^4 |\ln z|\right),$$

where γ is the Euler-Mascheroni constant, and we see that at $d \ll l_c$ the contribution of massive modes to the Casimir energy, as it follows from the qualitative considerations, is comparable to the contribution of a massless field.

Graph of the dependence of the Casimir energy

$$\mathcal{E}_{cas} = -\frac{4Z\mu_1\mu_2}{15d^2}\mathcal{F}(2md) \quad (38)$$

versus the distance between the interacting strings on a double logarithmic scale is shown in Fig. 2. The dashed line corresponds to the massless limit.

4. CONCLUSION

In the framework of the $\mathrm{tr}\ln$-formalism, the vacuum interaction of cosmic strings was considered in the approximation, when their transverse size can be ignored, but the mass of the quantized field is not assumed to be zero. The main result is that at distances less than the Compton length, but noticeably exceeding the radius of the strings, the partial contribution of massive fields to the energy of the Casimir interaction of strings is comparable to the contribution of a massless field. Thus, at small distances, in this sense, the mass in the first approximation can be neglected. However, if this distance can no longer be considered large compared to the transverse size of the strings, then it is no longer possible to neglect the radius of the strings. In this case, we again have two parameters with the same dimension, but in the present case these ones are the radius of the strings a and the distance between them d. As a result, the evaluation formula (1) is replaced by

$$\frac{\mathcal{E}_{cas}}{Z} = -\frac{4}{15\pi}\frac{\mu_1\mu_2}{d^2}\Phi\left(\frac{a}{d}\right). \quad (39)$$

It follows that the scale on which the transverse size of the strings is affected, is their radius. As in the case discussed in the Introduction, for $\tilde{z} = a/d \to 0$ the function $\Phi(\tilde{z})$ tends to one. Indeed, if this limit is defined as a limit transition $d \to \infty$, it is precisely obvious that at such distances the strings must interact as infinitely thin. Therefore, the result must coincide with the energy of the interaction of two infinitely thin strings, i.e. with the coefficient at Φ. The answer should be the same at a tending to zero, but in the case of thick strings $d \geqslant 2a$. Therefore, a noticeable difference of Φ from unity and, consequently, a noticeable dependence of the Casimir energy on the transverse size of the strings will occur if the distance between the strings does not much exceed $2a$. In the work [25] we have shown that this is indeed the case. Moreover, at these distances, the vacuum interaction energy of thick strings can even noticeably exceed a similar value for infinitely thin strings with the same mass per unit length.

The results obtained may be useful in studying the interaction of strings during near-collision, their collision and their entanglement and reconnection.

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REFERENCES

1. The Formation and Evolution of Cosmic Strings, ed. by G.W.Gibbons, S. W.Hawking, and T.Vachaspati, Cambridge Univ. Press, Cambridge (1990).

2. A.Vilenkin and E.P. S. Shellard, Cosmic Strings and Other Topological Defects, Cambridge Univ. Press, Cambridge (1994).

3. M.Bordag, Ann. Phys. (Berlin) 47, 93 (1990).

4. D.V.Gal'tsov, Yu.V.Grats, and A.B.Lavrent'ev, Phys. Atom. Nucl. 58, 516 (1995)

5. J.M.Muñoz-Castañeda and M.Bordag, Phys. Rev. D 89, 065034 (2014).

6. Yu.V.Grats, Theor. Math. Phys. 186, 205 (2016).

7. P.C. Letelier, Class. Quant. Grav. 4, L75 (1987).

8. T.M.Helliwell and D.A.Konkowski, Phys. Rev. D 34, 1918 (1986).

9. V.P. Frolov and E.M. Serebriany, Phys. Rev. D 35, 3779 (1987).

10. B. Linet, Phys. Rev. D 35, 536 (1987).

11. J. S.Dowker, Phys.Rev.D 36, 3095 (1987).

12. J. S.Dowker, Phys.Rev.D 36, 3742 (1987).

13. Y.V.Grats and P. Spirin, Eur.Phys. J.C 77, 101 (2017).

14. Y.V.Grats and P. Spirin, Int. J.Mod.Phys.A 35, 2040030 (2020).

15. B. S.De Witt, Phys.Rep.C 19, 297 (1975).

16. B. S. DeWitt, Dynamical Theory of Groups and Fields, Gordon and Breach, New York (1965)

17. N.D.Birrell and P.C.W.Davies, Quantum Fields in Curved Space, Cambridge Univ. Press, Cambridge (1982)

18. M.E.Peskin and D.V. Schroeder, An Introduction to Quantum Field Theory, Addison-Wesley, Reading (1995)

19. M.Bordag, G. L.Klimchitskaya, U. Mohideen, and V.M. Mostepanenko, Advances in the Casimir Effect, Oxford Univ. Press, Oxford (2009).

20. M.Reed and B. Simon, Methods of Mathematical Physics III: Scattering Theory, Academic Press, New York (1979)

21. C. Itzykson and J.B. Zuber, Quantum Field Theory, McGraw-Hill (1980)

22. I.M.Gel'fand and G.E.Shilov, Generalized Functions: Properties and Operations, Academic Press, Waltham (1964)

23. A.P.Prudnikov, Yu.Brychkov, and O.Marichev, Integrals and Series, Vol. 2, Special Functions, Gordon and Breach Science Publishers, New York (1990)

24. H.Bateman and A.Erdelyi, Higher Transcendental Functions, McGraw-Hill (1953)

25. Y.V.Grats and P. Spirin, Phys. Rev.D 108, 045001 (2023)